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Approximate Solution for Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operator

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Abstract
In this paper, the nonlinear gas dynamic and coupled KdV equations within local fractional operator are discussed. The approximate solutions are obtained by using the local fractional variational iteration method (LFVIM). This method is able to solve large class of linear and nonlinear equations effectively, more easily and accurately; and thus the method has been widely applicable to solve any class of equations in sciences and engineering.

Introduction

Many phenomena in biology, fluid flow, physical problems and other sciences can be described very successfully by nonlinear models. The equations of gas dynamics are mathematical expressions based on the physical laws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy, and so forth. The nonlinear equations of ideal gas dynamics are applicable for three types of nonlinear waves like shock fronts, rare factions, and contact discontinuities *[**1**]*. The Korteweg-de Vries Equation (KdV equation) describes the theory of water waves in shallow channels. It is a non-linear equation which exhibits special solutions, known as Solitons, which are stable and do not disperse with time.

There are many analytical and numerical methods used to solve local fractional partial differential equations such as, local fractional function decomposition method *[**2,3**]*, local fractional Adomian decomposition method *[**3,4**]*, local fractional series expansion method *[**5,6**]*, local fractional Laplace transform method *[**7,8**]*, local fractional Fourier series method *[**9**]*, local fractional Laplace decomposition method *[**10,11**]*, local fractional Laplace variational iteration method *[**12,13,14**]*, and another methods. The rest of this paper is organized as follows. In Section 2, basic ideas of the local fractional variational iteration method is presented. In Section 3, the application of LFVIM for solving nonlinear gas dynamic equation and coupled KdV is presented. Conclusion will appear in the last section.

Analysis of the Local Fractional Variational Iteration Method

To illustrate the basic concept of the local fractional variational Iteration method, we consider the following general nonlinear local fractional partial differential equation:

$$L_\alpha u(x,y)+R_\alpha u(x,y)+N_\alpha u(x,y)=f(x,y)\ ,0<\alpha\leq 1\,, \tag{1}$$

where $L_\alpha=\dfrac{\partial^\alpha}{\partial x^\alpha}$, R_α denotes linear local fractional derivative operator, N_α denotes nonlinear local fractional derivative operator, and $f(x,y)$ is the nondifferentiable source term.

According to the rule of local fractional variational iteration method, the correction local fractional functional for Eq. (1) is constructed as [15,16]:

$$u_{n+1}(x,y)=u_n(x,y)+{}_0I_x^{(\alpha)}\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi,y)+R_\alpha\tilde{u}_n(\xi,y)+N_\alpha\tilde{u}_n(\xi,y)-f(\xi,y)\right]\right), \tag{2}$$

where $\dfrac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}$ is a fractal Lagrange multiplier, $\tilde{u}_n$ is considered as a restricted local fractional variation and the local fractional operator be defined as

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha, \tag{3}$$

with the partition of the interval $[a,b]$ is denoted as (t_j,t_{j+1}), $j=0,...,N-1$, $t_0=a$ and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,.....\}$.

Making the local fractional variation of Eq. (2), we have

$$\delta^\alpha u_{n+1}(x,y)=\delta^\alpha u_n(x,y)+{}_0I_x^{(\alpha)}\delta^\alpha\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi,y)+R_\alpha\tilde{u}_n(\xi,y)+N_\alpha\tilde{u}_n(\xi,y)-f(\xi,y)\right]\right). \tag{4}$$

The extremum condition of $u_{n+1}(x,y)$ is given by

$$\delta^\alpha u_{n+1}=0\,. \tag{5}$$

In view of (5), we have the followingstationary conditions:

$$1+\left.\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\right|_{\xi=x}=0\,,\quad \left.\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=x}=0. \tag{6}$$

This in turn gives

$$\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}=-1, \tag{7}$$

so that iteration is expressed as

$$u_{n+1}(x,y)=u_n(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha u_n(\xi,y)}{\partial\xi^\alpha}+R_\alpha u_n(\xi,y)+N_\alpha u_n(\xi,y)-f(\xi,y)\right)(d\xi)^\alpha\,,\ n\geq 0. \tag{8}$$

Finally, from (8), we obtain the solution of (1) as follows:

$$u(x,y)=\lim_{n\to\infty}u_n(x,y). \tag{9}$$

The above formula plays an important role in dealing with the local fractional differential equation with either linearity or nonlinearity.

Some Illustrative Examples

In this section, we given some illustrative examples for solving the nonlinear gas dynamic and coupled KdV equations within local fractional derivative operator by using local fractional variational iteration method.

Example 1. Let us consider the following nonlinear gas dynamics equation involving local fractional derivative operator:

$$\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u^{2}(x,y)}{\partial y^{\alpha}}-u(x,y)\big(1-u(x,y)\big)=-E_{\alpha}(x^{\alpha}-y^{\alpha}), \tag{10}$$

with the initial value conditions as follows:

$$u(0,y)=1-E_{\alpha}(-y^{\alpha}). \tag{11}$$

By using (8), we have a local fractional iteration procedure as

$$u_{n+1}(x,y)=u_n(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}u_n(\xi,y)}{\partial\xi^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u_n^2(\xi,y)}{\partial y^{\alpha}}-u_n(\xi,y)\big(1-u_n(\xi,y)\big)+E_{\alpha}(\xi^{\alpha}-y^{\alpha})\right)(d\xi)^{\alpha},\ n\geq 0. \tag{12}$$

Suppose that an initial approximation has the following form which satisfies the initial condition:

$$u_0(x,y)=1-E_{\alpha}(-y^{\alpha}). \tag{13}$$

Now by (12), we obtain the following approximations:

$$u_1(x,y)=u_0(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}u_0(\xi,y)}{\partial\xi^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u_0^2(\xi,y)}{\partial y^{\alpha}}-u_0(\xi,y)\big(1-u_0(\xi,y)\big)+E_{\alpha}(\xi^{\alpha}-y^{\alpha})\right)(d\xi)^{\alpha}$$

$$=1-E_{\alpha}(-y^{\alpha})-\frac{1}{\Gamma(1+\alpha)}\int_0^x\Big(E_{\alpha}(\xi^{\alpha}+y^{\alpha})\Big)(d\xi)^{\alpha}$$

$$=1-E_{\alpha}(-y^{\alpha})-E_{\alpha}(x^{\alpha}-y^{\alpha})+E_{\alpha}(-y^{\alpha})$$

$$=1-E_{\alpha}(x^{\alpha}-y^{\alpha}), \tag{14}$$

$$u_2(x,y)=u_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}u_1(\xi,y)}{\partial\xi^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u_1^2(\xi,y)}{\partial y^{\alpha}}-u_1(\xi,y)\big(1-u_1(\xi,y)\big)+E_{\alpha}(\xi^{\alpha}-y^{\alpha})\right)(d\xi)^{\alpha}$$

$$=1-E_{\alpha}(-y^{\alpha})-\frac{1}{\Gamma(1+\alpha)}\int_0^x\Big(E_{\alpha}(\xi^{\alpha}+y^{\alpha})\Big)(d\xi)^{\alpha}$$

$$=1-E_{\alpha}(x^{\alpha}-y^{\alpha}), \tag{15}$$

……………………………….

$$u_n(x,y)=u_{n-1}(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}u_{n-1}(\xi,y)}{\partial\xi^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u_{n-1}^2(\xi,y)}{\partial y^{\alpha}}-u_{n-1}(\xi,y)\big(1-u_{n-1}(\xi,y)\big)+E_{\alpha}(\xi^{\alpha}-y^{\alpha})\right)(d\xi)^{\alpha}$$

$$=1-E_{\alpha}(x^{\alpha}-y^{\alpha}). \tag{16}$$

This gives the exact solution by

$$u(x,y)=\lim_{n\to\infty}u_n(x,y)$$

$$=1-E_{\alpha}(x^{\alpha}-y^{\alpha}). \tag{17}$$

Example 2. Consider the coupled KdV equations with local fractional derivative:

$$\begin{aligned}&\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u(x,y)}{\partial y^{3\alpha}}+2u(x,y)\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}+2v(x,y)\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}=0,\\&\frac{\partial^{\alpha}v(x,y)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}v(x,y)}{\partial y^{3\alpha}}+2v(x,y)\frac{\partial^{\alpha}v(x,y)}{\partial y^{\alpha}}+2u(x,y)\frac{\partial^{\alpha}v(x,y)}{\partial y^{\alpha}}=0,\end{aligned}\tag{18}$$

subject to the initial conditions

$$\begin{aligned}&u(0,y)=E_{\alpha}(-y^{\alpha}),\\&v(0,y)=-E_{\alpha}(-y^{\alpha}).\end{aligned}\tag{19}$$

To solve the system of equations (18) by means of the local fractional variational iteration method, we construct the following correction functional:

$$\begin{aligned}&u_{m+1}(x,y)=u_m(x,y)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{\lambda_1(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}\tilde{u}_m}{\partial y^{3\alpha}}+2u_m\frac{\partial^{\alpha}\tilde{u}_m}{\partial y^{\alpha}}+2v_m\frac{\partial^{\alpha}\tilde{u}_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha},\\&v_{m+1}(x,y)=v_m(x,y)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{\lambda_2(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\partial^{\alpha}v_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}\tilde{v}_m}{\partial y^{3\alpha}}+2v_m\frac{\partial^{\alpha}\tilde{v}_m}{\partial y^{\alpha}}+2u_m\frac{\partial^{\alpha}\tilde{v}_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha},\end{aligned}\tag{20}$$

where $\frac{\lambda_i(\xi)^{\alpha}}{\Gamma(1+\alpha)}$ $,i=1,2$ are fractal Lagrange multipliers.

Taking the local fractional variation of (20), we have

$$\begin{aligned}&\delta^{\alpha}u_{m+1}(x,y)=\delta^{\alpha}u_m(x,y)+\frac{\delta^{\alpha}}{\Gamma(1+\alpha)}\int_0^x\frac{\lambda_1(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}\tilde{u}_m}{\partial y^{3\alpha}}+2u_m\frac{\partial^{\alpha}\tilde{u}_m}{\partial y^{\alpha}}+2v_m\frac{\partial^{\alpha}\tilde{u}_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha},\\&\delta^{\alpha}v_{m+1}(x,y)=\delta^{\alpha}v_m(x,y)+\frac{\delta^{\alpha}}{\Gamma(1+\alpha)}\int_0^x\frac{\lambda_2(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\partial^{\alpha}v_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}\tilde{v}_m}{\partial y^{3\alpha}}+2v_m\frac{\partial^{\alpha}\tilde{v}_m}{\partial y^{\alpha}}+2u_m\frac{\partial^{\alpha}\tilde{v}_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha}.\end{aligned}\tag{21}$$

The extremum conditions of (21) is given by

$$\begin{aligned}&\delta^{\alpha}u_{m+1}=0,\\&\delta^{\alpha}v_{m+1}=0.\end{aligned}\tag{22}$$

In view of (22), we have the following stationary conditions:

$$1+\left.\frac{\lambda_1(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right|_{\xi=x}=0,\ \left.\left(\frac{\lambda_1(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=x}=0,\ \&\quad 1+\left.\frac{\lambda_2(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right|_{\xi=x}=0,\ \left.\left(\frac{\lambda_2(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=x}=0.\tag{23}$$

This in turn gives

$$\frac{\lambda_i(x)^{\alpha}}{\Gamma(1+\alpha)}=-1,\ i=1,2.\tag{24}$$

so that iteration is expressed as

$$\begin{aligned}&u_{m+1}(x,y)=u_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}u_m}{\partial y^{3\alpha}}+2u_m\frac{\partial^{\alpha}u_m}{\partial y^{\alpha}}+2v_m\frac{\partial^{\alpha}u_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha},\\&v_{m+1}(x,y)=v_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}v_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{3\alpha}v_m}{\partial y^{3\alpha}}+2v_m\frac{\partial^{\alpha}v_m}{\partial y^{\alpha}}+2u_m\frac{\partial^{\alpha}v_m}{\partial y^{\alpha}}\right)(d\xi)^{\alpha},m\geq 0.\end{aligned}\tag{25}$$

Suppose that an initial approximation has the following form which satisfies the initial condition:

$$\begin{aligned}&u_0(x,y)=E_{\alpha}(-y^{\alpha}),\\&v_0(x,y)=-E_{\alpha}(-y^{\alpha}).\end{aligned}\tag{26}$$

Now by iteration formula (25), we obtain the following approximations:

$$u_1(x,y)=u_0(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha u_0(\xi,y)}{\partial\xi^\alpha}+\frac{\partial^{3\alpha}u_0(\xi,y)}{\partial y^{3\alpha}}+2u_0(\xi,y)\frac{\partial^\alpha u_0(\xi,y)}{\partial y^\alpha}+2v_0(\xi,y)\frac{\partial^\alpha u_0(\xi,y)}{\partial y^\alpha}\right)(d\xi)^\alpha$$

$$v_1(x,y)=v_0(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha v_0(\xi,y)}{\partial\xi^\alpha}+\frac{\partial^{3\alpha}v_0(\xi,y)}{\partial y^{3\alpha}}+2v_0(\xi,y)\frac{\partial^\alpha v_0(\xi,y)}{\partial y^\alpha}+2u_0(\xi,y)\frac{\partial^\alpha v_0(\xi,y)}{\partial y^\alpha}\right)(d\xi)^\alpha$$

$$=E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(-E_\alpha(-y^\alpha)-2E_\alpha(-y^\alpha)+2E_\alpha(-y^\alpha)\right)(d\xi)^\alpha=E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right),$$

$$=-E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(E_\alpha(-y^\alpha)-2E_\alpha(-y^\alpha)+2E_\alpha(-y^\alpha)\right)(d\xi)^\alpha=-E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}\right), \quad (27)$$

$$u_2(x,y)=u_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha u_1(\xi,y)}{\partial\xi^\alpha}+\frac{\partial^{3\alpha}u_1(\xi,y)}{\partial y^{3\alpha}}+2u_1(\xi,y)\frac{\partial^\alpha u_1(\xi,y)}{\partial y^\alpha}+2v_1(\xi,y)\frac{\partial^\alpha u_1(\xi,y)}{\partial y^\alpha}\right)(d\xi)^\alpha$$

$$v_2(x,y)=v_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha v_1(\xi,y)}{\partial\xi^\alpha}+\frac{\partial^{3\alpha}v_1(\xi,y)}{\partial y^{3\alpha}}+2v_1(\xi,y)\frac{\partial^\alpha v_1(\xi,y)}{\partial y^\alpha}+2u_1(\xi,y)\frac{\partial^\alpha v_1(\xi,y)}{\partial y^\alpha}\right)(d\xi)^\alpha$$

$$=E_\alpha(-y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(-\frac{\xi^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right)(d\xi)^\alpha$$

$$=-E_\alpha(-y^\alpha)-\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right)(d\xi)^\alpha$$

$$=E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right),$$

$$=-E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right), \quad (28)$$

……………………………….

$$u_m(x,y)=E_\alpha(-y^\alpha)\sum_{k=0}^{m}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)},$$

$$v_m(x,y)=-E_\alpha(-y^\alpha)\sum_{k=0}^{m}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}. \quad (29)$$

Therefore, the series solutions can be written in the form

$$u(x,y)=E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right),$$

$$v(x,y)=-E_\alpha(-y^\alpha)\left(1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right), \quad (30)$$

and finally in its closed form gives

$$u(x,y)=E_\alpha(x^\alpha-y^\alpha),$$

$$v(x,y)=-E_\alpha(x^\alpha-y^\alpha). \quad (31)$$

Conclusions

In this paper, the local fractional variational iteration method has been successfully applied to finding the approximate analytical solutions of nonlinear gas dynamic and coupled KdV equations with local fractional operator. The solution obtained by the local fractional variational iteration method is an infinite power series for appropriate initial condition, which can, in turn, be expressed in a closed form, the exact solution. The results show that this method is a powerful mathematical tool for solving nonlinear gas dynamic and coupled KdV equations, it is also a promising method to solve other nonlinear equations.

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References

[1] H. Aminikhah and A. Jamalian, *Numerical Approximation for Nonlinear Gas Dynamic Equation*, International Journal of Partial Differential Equations, vol. 2013, Article ID 846749, pp. 1-7, (2013).

[2] S. Q. Wang, Y. J. Yang and H. K. Jassim, *Local fractional function decomposition method for solving inhomogeneous wave equations with local fractional derivative*, Abstract and Applied Analysis, Article ID 176395, pp. 1-7, (2014).

[3] S. P. Yan, H. Jafari and H. K. Jassim, *Local fractional Adomian decomposition and function decomposition methods for solving Laplace equation within local fractional operators*, Advances in Mathematical Physics, Article ID 161580, pp. 1-7, (2014).

[4] D. Baleanu, J.A.T. Machado, C. Cattani, M. C. Baleanu and X.J. Yang, *Local fractional variational iteration and decomposition methods for wave equation on Cantor sets*, Abstract and Applied Analysis, Article ID 535048, pp.1-6, (2014).

[5] H. Jafari, and H. K. Jassim, *Local Fractional Series Expansion Method for Solving Laplace and Schrodinger Equations on Cantor Sets within Local Fractional Operators*, International Journal of Mathematics and Computer Research, Vol. 2, No. 11 ,736-744, (2014).

[6] A. M. Yang, Z. S. Chen, X. J. Yang, *Application of the Local Fractional Series Expansion Method and the Variational Iteration Method to the Helmholtz Equation Involving Local Fractional Operators*, Abstract and Applied Analysis, Article ID 259125, pp. 1-6, (2013).

[7] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, NY, USA, (2012).

[8] X. J. Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic, Hong Kong, China, (2011).

[9] M. S. Hu, R. P. Agarwal, and X. J. Yang, *Local fractional Fourier series with application to wave equation in fractal vibrating string*, Abstract and Applied Analysis, Article ID 567401, pp. 1-15, (2012).

[10] H. Jafari, H. K. Jassim, *Numerical Solutions of Telegraph and Laplace Equations on Cantor Sets Using Local Fractional Laplace Decomposition Method* , International Journal of Advances in Applied Mathematics and Mechanics, Vol. 2, No. 3, pp. 1-8, (2015).

[11] H. K. Jassim, *Local Fractional Laplace Decomposition Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow with Local Fractional Derivative*, International Journal of Advances in Applied Mathematics and Mechanics, Vol. 2, No. 7, pp. 1-7, (2015).

[12] H. Jafari, and H. K. Jassim, *Local Fractional Laplace Variational Iteration Method for Solving Nonlinear Partial Differential Equations on Cantor Sets within Local Fractional Operators*, Journal of Zankoy Sulaimani-Part A, vol. 16, no. 4, pp. 49-57, (2014).

[13] H. K. Jassim, C. Ünlü, S. P. Moshokoa, C. M. Khalique, *Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators*, Mathematical Problems in Engineering, Vol. 2015, Article ID 309870, pp. 1-7, (2015).

[14] C. F. Liu, S. S. Kong, and S. J. Yuan, *Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem*, Thermal Science, vol. 17, no. 3, pp. 715–721, (2013).

[15] X. J. Yang and D. Baleanu, *Local fractional variational iteration method for Fokker-Planck equation on a Cantor set*, Acta Universitaria, Vol. 23, No. 2, pp. 3-8, (2013).

[16] X. J. Yang and D. Baleanu, *Fractal heat conduction problem solved by local fractional variation iteration method*. Thermal Science. Vol. 17, no. 2, pp. 625–628, (2013).